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Convection in a box: on the dependence of preferred wave-number upon the Rayleigh number at finite amplitude

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Using Stuart's shape assumption and a condition of maximum heat transport it is found that the preferred number of finite roll cells present in Bénard convection in a *three-dimensional rectangular box* tends to *decrease* with increasing supercritical Rayleigh number in contradiction to the behaviour in an infinite layer but in accordance with experimental observation.

This 'end effect' might explain the similar observation of wave-number decrease in the Taylor instability between rotating cylinders.

1. Introduction

When a horizontal layer of fluid is heated from below to its critical Rayleigh number, convection begins. In the laboratory where a layer is of necessity *bounded in horizontal extent*, Koschmieder (1966) has found that the preferred mode of convection is some number of finite roll cells, the shape and orientation being dependent upon the shape of the container. If the Rayleigh number is raised further, the wave-number of the rolls tends to *decrease*. This last observation has been made by Krishnamurti (1967) as well as by Koschmieder. A similar observation has been made by Coles (1965) in the related problem of the Taylor instability of fluid contained between concentric rotating cylinders where the wave-number of the Taylor vortices tends to decrease with increasing supercritical Taylor number.

Schlüter, Lortz & Busse (1965) have computed the effects of finite amplitude convection with regard to the preferred mode in the case of a fluid layer of *infinite horizontal extent*. They find that the wave-number preferred in finite amplitude convection *increases* with increasing supercritical Rayleigh number, i.e. the opposite dependence from that observed is found.

The object of this note is to show that the wave-number dependence on the Rayleigh number can be dominated by the presence of the lateral walls.

Although a full perturbation analysis for small but finite amplitude is desirable, the computational difficulties prompt us to use an approximation to this. We will use a slight generalization to Stuart's (1958) shape assumption using the approximate eigenfunctions of the linearized problem as given by Davis (1967) for a rectangular box. In all further references we denote this paper by D. In addition we assume that the preferred mode is that one which convects the most heat. All assumptions are discussed in detail.

The result is that when an array of K finite roll cells is preferred in linear theory over M finite roll cells, a transition at a supercritical Rayleigh number is indicated if $K > M$. When $K < M$, no such transition is indicated, i.e. the wave-number tends to *decrease* with increasing Rayleigh number in accordance with observation.

2. Analysis

The equations which govern a steady-state disturbance to the static, conductive initial state $\mathbf{v} \equiv 0$, $\overline{T}^{(0)} = -z$ are the following:

$$\Delta\mathbf{v} - \nabla p + \mathscr{R}^{\frac{1}{2}}\theta\mathbf{k} = \mathscr{P}^{-1}\mathscr{R}^{\frac{1}{2}}\mathbf{v}\cdot\nabla\mathbf{v}, \tag{2.1}$$

$$\Delta\theta - \mathscr{R}^{\frac{1}{2}}\overline{T}_z w = \mathscr{R}^{\frac{1}{2}}[\mathbf{v}\cdot\nabla\theta - (\overline{w\theta})_z], \tag{2.2}$$

$$\overline{T}_z = -1 + \mathscr{R}^{\frac{1}{2}}[(\overline{w\theta}) - \langle w\theta\rangle], \tag{2.3}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2.4}$$

where $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$, ∇ is the grad operator, θ is the departure from the mean temperature $\overline{T}$, a bar denotes an average over the x, y expanse of the box, subscripts denote partial derivatives and $\langle\ \rangle$ denotes an integral over the full volume. The scales used for non-dimensionalization are $q = (\alpha(\Delta T)gd\kappa/\nu)^{\frac{1}{2}}$, $\Delta T, \rho_0\nu q/d$ and depth d for velocity, temperature, pressure and length respectively. Here α, g, κ, ν and ρ_0 are the volume expansion coefficient, acceleration of gravity, thermal conductivity, kinematic viscosity and density at the centre of the layer respectively. ΔT is the over-all vertical temperature difference,

$$\mathscr{R} = \alpha(\Delta T)gd^3/\kappa\nu$$

is the Rayleigh number and $\mathscr{P} = \nu/\kappa$ is the Prandtl number. The boundaries are assumed to be rigid, perfectly heat-conducting planes and hence

$$\mathbf{v} = \theta = 0 \quad \text{on} \quad |x| = \tfrac{1}{2}h_1,\ |y| = \tfrac{1}{2}h_2,\ |z| = \tfrac{1}{2}. \tag{2.5}$$

We form an energy-type equation by taking the inner product of (2.1) with $\mathbf{v}$, multiplying (2.2) by θ, adding and integrating over the volume

$$\langle\nabla\mathbf{v}:\nabla\mathbf{v} + \nabla\theta\cdot\nabla\theta\rangle = 2\mathscr{R}^{\frac{1}{2}}\langle w\theta\rangle - \mathscr{R}\langle(\overline{w\theta})^2\rangle + \mathscr{R}\langle w\theta\rangle^2, \tag{2.6}$$

where we have used equations (2.3), (2.4), (2.5) and integration by parts.

We define the *convective heat transport* H by

$$H = \mathscr{R}^{\frac{1}{2}}\langle w\theta\rangle. \tag{2.7}$$

Using (2.7), equation (2.6) can be simplified to yield

$$H = m(1 - I\mathscr{R}^{-\frac{1}{2}}), \tag{2.8}$$

where
$$I(\mathbf{v},\theta) = \langle\nabla\mathbf{v}:\nabla\mathbf{v} + \nabla\theta\cdot\nabla\theta\rangle/2\langle w\theta\rangle \tag{2.9}$$

and
$$m = \frac{2\langle w\theta\rangle^2}{\langle(\overline{w\theta})^2\rangle - \langle w\theta\rangle^2}. \tag{2.10}$$

Note that m is insensitive to changes of scale in w and θ and hence to our choice of non-dimensional variables.

Let us now employ Stuart's (1958) shape assumption. The procedure is to evaluate (2.8) with $\mathbf{v} = \mathbf{v}^{(1)}$, $\theta = \theta^{(1)}$, the eigenfunctions of the linear theory at zero growth rate (marginal case). If we do this, the resulting expression involves $I(\mathbf{v}^{(1)}, \theta^{(1)})$. Since by equation (3.2) of D,

$$\mathscr{R}_c^{-\frac{1}{2}} = \max_S I^{-1},$$

where S is an appropriate function space, $I(\mathbf{v}^{(1)}, \theta^{(1)}) \equiv \mathscr{R}_c^{\frac{1}{2}}$. Therefore, the result of applying the shape assumption to (2.8)† is

$$H_{sa} = m_{sa}\left(1 - \frac{\mathscr{R}_c^{\frac{1}{2}}}{\mathscr{R}^{\frac{1}{2}}}\right), \tag{2.11}$$

where a subscript *sa* denotes the corresponding functional evaluated with the $(\mathbf{v}, \theta)$ of linear theory at zero growth rate.

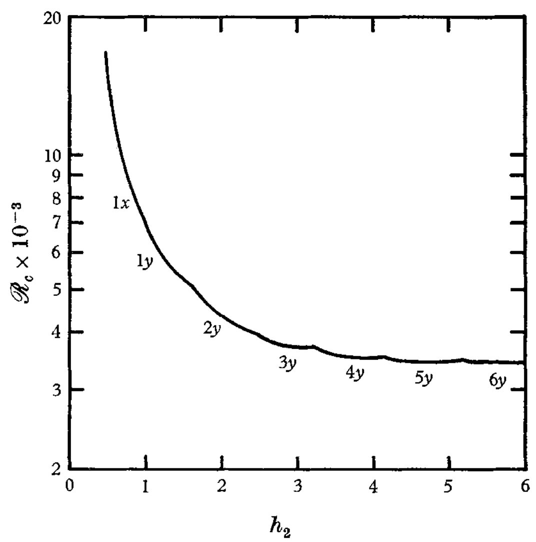

FIGURE 1. The dependence of $\mathscr{R}_c$ on h_2 for $h_1 = 1{\cdot}0$ according to linear theory. (Taken from figure 7 of Davis (1967).) m_x and n_y denote m finite x-rolls and n finite y-rolls respectively.

Let us now apply equation (2.11) to a rectangular box with lateral dimensions, say, $h_1 = 1{\cdot}0$ and variable h_2. Figure 1 (figure 7 of D) exhibits the dependence of $\mathscr{R}_c$ upon h_2. At $h_2 = h_{20} = 4{\cdot}15$, the independent stability curves of 4 finite y-rolls and 5 finite y-rolls intersect, the former being the preferred mode for linear theory in an interval of $h_2 < h_{20}$ and the latter in an interval of $h_2 > h_{20}$. The results of using the eigenfunctions from D for computing m_{sa} when $h_2 = 4{\cdot}0$ and $h_2 = 4{\cdot}5$ for both 4 and 5 finite y-rolls are given in table 1 along with the corresponding critical Rayleigh numbers. The plots of H_{sa} *vs.* $\mathscr{R}$ are given in these cases in figure 2.

† Equation (2.11) is valid to order $O(\mathscr{R} - \mathscr{R}_c)$ and is the analogue of equation (5.13) of Davey (1962) which the author suggests has the greatest validity of all equivalent formulae. Equation (2.11) in addition, reproduces the results of Schlüter *et al.* in the case of the layer of infinite horizontal extent.

Let us now *assume* that the preferred mode of convection at $\mathscr{R} > \mathscr{R}_c$ is that one which convects the most heat, i.e. the one with maximum H. Under this proviso, as $\mathscr{R}-\mathscr{R}_c$ is increased from zero, the solid curve of figure 2*a* is followed until $\mathscr{R} = \mathscr{R}^\dagger$. After this point, the dashed curve corresponds to a mode convecting

h_1	h_2	wave-number	$\mathscr{R}_c$	m_{sa}	$\mathscr{R}^\dagger$
1·0	4·5	5	3,482·8	1·386	11,408·4
1·0	4·5	4	3,626·4	1·422	
1·0	4·0	5	3,646·1	1·376	—
1·0	4·0	4	3,535·0	1·411	

TABLE 1.

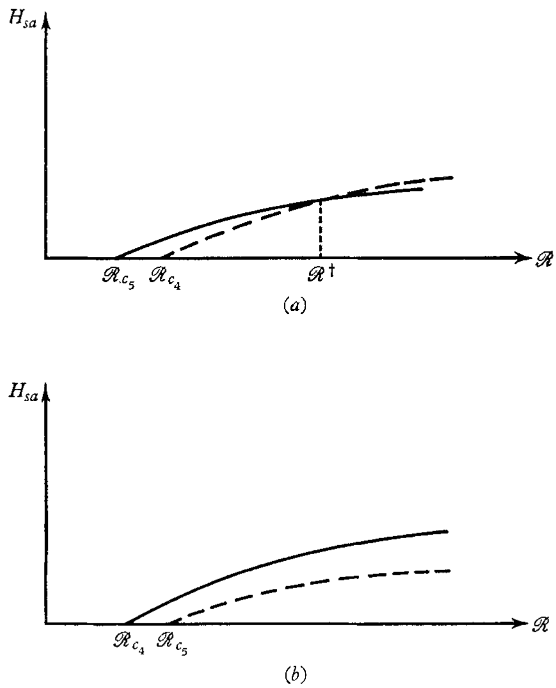

FIGURE 2 (*a*). An indicated transition. Convective heat transfer using the shape assumption as a function of $\mathscr{R}$. The solid curve corresponds to 5 finite y-rolls, the dashed curve to 4 finite y-rolls. $h_1 = 1{\cdot}0$, $h_2 = 4{\cdot}5$. (*b*) No indicated transition. Convective heat transfer using the shape assumption as a function of $\mathscr{R}$. The solid curve corresponds to 4 finite y-rolls, the dashed curve to 5 finite y-rolls. $h_1 = 1{\cdot}0$, $h_2 = 4{\cdot}0$.

more heat and hence a transition from 5 finite y-rolls to 4 finite y-rolls is indicated. In figure 2*b*, however, the two curves, rather than intersecting, tend to diverge with increasing $\mathscr{R}-\mathscr{R}_c$. The maximum H then always corresponds to 4 finite y-rolls. No transition is indicated.

The above behaviour is typical of situations for all of the h_1 and h_2 examined in the range $0{\cdot}25 \leqslant h_1, h_2 \leqslant 6$ covered by D. In the above, we say that the transition is only *indicated* (rather than assured) because the shape assumption is only quantitatively correct in the interval $\mathscr{R}_c \leqslant \mathscr{R} \leqslant 1{\cdot}1\mathscr{R}_c$. In particular, Davey (1962, figures 3 and 4) has found in the axi-symmetric, small gap Taylor problem near solid body rotation that the shape assumption yielded the torque needed to maintain constant rotation (analogous to H) to within 1% of that predicted by a formal perturbation procedure in the Taylor number interval $(T_c, 1{\cdot}1\ T_c)$, within 3% up to $2T_c$ and within about 10% up to $4T_c$.

3. Discussion and conclusions

The shape assumption was suggested by Stuart (1958) for the calculation of the finite equilibrium amplitude attained by a disturbance to the unstable flow between concentric rotating cylinders. The method neglects the generation of harmonics of the fundamental (linear theory solution) and the subsequent distortion of the fundamental and mean flow by the harmonics. Davey (1962) has used a formal perturbation analysis in the calculation of this secondary flow. He has found that the equilibrium amplitude attained near the critical Taylor number is precisely that which would be obtained using the shape assumption. Since the equations governing an axi-symmetric unstable disturbance to the flow between rotating cylinders with small spacing and in nearly solid body rotation are precisely those governing the thermal convection of an infinite layer with the Prandtl number equal to one, application of the shape assumption to convection in a box seems valid. Although valid, the shape assumption cannot distinguish a stable disturbance from an unstable one. Hence we *assume* that at finite amplitude finite rolls are the preferred mode of convection. They are observed in experiment and are preferred at infinitesimal amplitude (Davis 1967). Schlüter, *et al.* (1965) have shown by a formal perturbation analysis that in the infinite layer with constant properties the preferred mode is that one convecting the most heat. We have therefore taken this as our criterion for selection. It is well to note that had Schlüter *et al.* used the shape assumption and the stability criterion of maximum H, their results would have been unaltered.

We summarize our conclusions as follows: when a rectangular box of fluid is heated from below beyond its critical Rayleigh number, there is a tendency for the number of finite rolls preferred to *decrease*. More precisely, if K finite rolls are preferred over M finite rolls in linear theory (for given dimensions of the box), at finite amplitude there is a transition indicated from K to M finite rolls if $K > M$ but none if $K < M$.

Coles (1965) has noted that when the Taylor number in Taylor instability is increased beyond its critical value, the axial wave-number of the vortices tends to *decrease*. Although the ends of the concentric cylinders cause the basic Couette flow to be altered making our model of convection in a box never strictly identical, our conclusions concerning the convective model might still indicate the cause of this behaviour.

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REFERENCES

COLES, D. 1965 *J. Fluid Mech.* **21**, 3.

DAVEY, A. 1962 *J. Fluid Mech.* **14**, 3.

DAVIS, S. H. 1967 *J. Fluid Mech.* **30**, 465.

KOSCHMIEDER, E. L. 1966 *Beitr. Phys. Atmos.* **39**, 1.

KRISHNAMURTI, R. 1967 Doctoral Dissertation, Institute of Geophysics and Planetary Physics, U.C.L.A., Los Angeles, California.

SCHLÜTER, A., LORTZ, D. & BUSSE, F. 1965 *J. Fluid Mech.* **23**, 1.

STUART, J. T. 1958 *J. Fluid Mech.* **4**, 1.